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APPLICATION OF THE PRINCIPLE OF CHOICE TO THE PROBLEM OF THE INITIAL DEVELOPMENT OF SLIP LINES FROM A CORNER POINT*

L.A. KIPNIS and G.P. CHEREPANOV

Symmetrical problems of initial development near the corner point of the boundary of the body of a plastic zone modelled by two straight slip lines emerging from the vertex are considered under conditions of plane deformation. Functional Wiener-Hopf equations of the problems and their exact analytic solutions are given. The length of the slip lines and the angle of their inclination to the boundary are determined. The principle of choice is used to find the latter. According to this principle, of all possible directions of the development of slip lines, the direction realized corresponds to the maximum value of the rate of dissipation of energy by the body.

The last few years have seen the publication of a number of papers in mechanics of fracture, dealing with problems of initial development within the bodies, near the concentrators, of plastic zones modelled by straight slip lines emerging from the vertex at some angle to the boundary /1-6/. Everyone of these problems reduces to a functional Wiener-Hopf equation, and its solution is used to establish the dependence of the length of the slip line on its angle of inclination to the boundary, the latter being a free parameter. The value of this angle at which the slip line is of maximum length, is taken as the unknown quantity which determines the direction in which the slip line develops.

In the present paper a new, stricter approach is proposed, towards solving the problem of the direction in which the slip lines emerging from the corner point develop. The approach is based on the principle of

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choice /7/. According to this principle, of all possible directions, the direction realised is that corresponding to the maximum value of the rate of dissipation of energy by the body.

1. We consider, under conditions of plane deformation, the problem of the initial development of plastic deformations near the corner point O of the boundary of a homogeneous, isotropic, ideally elastoplastic body (Fig.1). The problem is assumed to be symmetrical about the bisectrix of the angle. We study the cases with the following boundary conditions specified at its edges: a) the edges are rigidly clamped; b) the edges are stress-free; c) tangential stress and normal displacement at the edges are equal to zero. We assume that the plastic deformations are concentrated along two straight slip lines which emerge from the corner point and are symmetrical about the bisectrix, with the length of the slip lines small compared with the size of the body.

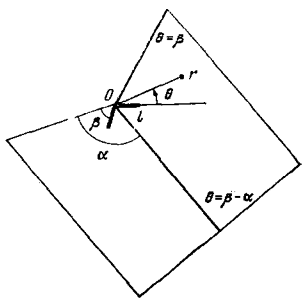

Fig.1

Using the "microscope principle" /8/ and taking into account the symmetry of the problem in question, we arrive, in each of the cases a), b) and c), at the corresponding boundary value problem of class N /8/ of the equilibrium of an infinite elastic wedge $\beta-\alpha<\theta<\beta,\ 0<r<\infty,\ \pi/2<\alpha<\pi$ containing a slip line at the vertex. An asymptotic curve is formed at infinity, representing a solution of the singular canonical problem of the theory of elasticity for a wedge /8, 9/ determined, apart from a single arbitrary real constant C, which characterizes the intensity of external field and which is assumed given.

We require to find the length l of the slip line and the angle β of its inclination to the edge.

Below we give the solutions of the problem listed above, of class N, and determine the function $l(\beta)$. According to the principle of choice we take, as the angle of inclination of the slip line to the edge, the value of $\beta\in\,]0;\alpha[$ which imparts the maximum value to the function

$$W(\beta)=\left|\tau_s\int_0^l\langle u_r^{\cdot}\rangle|_{\theta=0}\,dr\right|$$

Here τ_s is the limiting shear yield, $\langle a\rangle$ is the jump in the value of a, u_r, u_θ are the displacements and a dot denotes differentiation with respect to C.

In what follows, we assume that the given load parameter C increases and decreases with time (simple loading).

2. Let the case a) be realized. The boundary conditions have the form

$$\theta=\beta,\quad u_\theta=u_r=0;\quad \theta=\beta-\alpha,\quad \tau_{r\theta}=0,\quad u_\theta=0\ (\beta\in\,]0;\beta_0[\,\cup\,]\beta_0;\alpha[) \tag{2.1}$$

$$\theta=0,\ \langle\sigma_\theta\rangle=\langle\tau_{r\theta}\rangle=0,\ \langle u_\theta\rangle=0$$

$$\theta=0,\ r<l,\ \tau_{r\theta}=\tau_1;\ \theta=0,\ r>l,\ \langle u_r\rangle=0 \tag{2.2}$$

$$\theta=0,\quad r\to l-0,\quad \left\langle\frac{\partial u_r}{\partial r}\right\rangle\sim-\frac{4(1-\nu^2)}{E}\frac{k_{II}}{\sqrt{2\pi(l-r)}} \tag{2.3}$$

$$\theta=0,\quad r\to l+0,\quad \tau_{r\theta}\sim\frac{k_{II}}{\sqrt{2\pi(r-l)}}$$

$$\theta=0,\ r\to\infty,\ \tau_{r\theta}=Cgr^\lambda+o(1/r) \tag{2.4}$$

$$g=\frac{(2\pi)^\lambda}{2\varkappa}\left[\frac{(\lambda+1+\varkappa)\sin\lambda\alpha}{\sin(\lambda+2)\alpha}\sin(\lambda+2)(\alpha-\beta)-\lambda\sin\lambda(\alpha-\beta)\right]$$

$$(\varkappa=3-4\nu;\ g<0\ \text{when}\ \beta\in[0;\beta_0[;\ g>0\ \text{when}\ \beta\in\,]\beta_0;\alpha[;\ g(\beta_0)=0)$$

Here σ_θ, $\tau_{r\theta}$, σ_r are the stresses, E is Young's modulus, ν is Poisson's ratio, k_{II} is the stress intensity coefficient at the top of the slip line to be determined, and $\lambda\in\,]-1/2;0[$ is a unique root of the characteristic equation

$$\varkappa\sin2(\lambda+1)\alpha-(\lambda+1)\sin2\alpha=0$$

corresponding to the canonical singular problem in the interval $]-1;0[$; where C is the constant introduced in Sect.1, which has the dimension of force divided by length to the power $\lambda+2$; $\tau_1=\tau_s$, if $Cg>0$; $\tau_1=-\tau_s$, if $Cg<0$.

The solution of the formulated problem represents the sum of solutions of the following two problems. The first (problem A) differs from the original problem in that instead of the first condition (2.2) we have

$$\theta = 0,\ r < l,\ \tau_{r\theta} = \tau_1 - Cgr^{\lambda} \tag{2.5}$$

and at infinity the stresses decay as $o(1/r)$ (in the expression for $\tau_{r\theta}$ in (2.4) the first term is not present). The second problem represents the corresponding canonical singular problem for a wedge. Since the solution of the second problem is known, it remains to construct the solution of problem A.

Applying the integral Mellin transform with a complex parameter p /10/ to the equilibrium equations, to the compatibility of deformation conditions, to Hooke's law and to the conditions (2.1), and taking into account the second condition (2.2) and condition (2.5), we arrive at the functional Wiener-Hopf problem A

$$\frac{\tau_1}{p+1} + \frac{\tau_2}{p+\lambda+1} + \Phi^+(p) = K(p)G(p)\Phi^-(p)\,(-\varepsilon_1 < \operatorname{Re} p < \varepsilon_2) \tag{2.6}$$

$$\Phi^+(p) = \int_1^\infty \tau_{r\theta}(\rho l, 0)\rho^p\, d\rho, \quad \Phi^-(p) = \frac{E}{4(1-\nu^2)}\int_0^1 \left\langle \frac{\partial u_r}{\partial r}\right\rangle\Big|_{\substack{r=\rho l\\ \theta=0}} \rho^p\, d\rho$$

$$\tau_2 = -Cgl^{\lambda},\ K(p) = \operatorname{ctg} p\pi$$

$$G(p) = \frac{\{\delta[\varkappa_1 - 2(\varkappa\sin^2 p\beta + p^2\sin^2\beta)] - \Delta(\varkappa\sin 2p\beta - p\sin 2\beta)\}\sin p\pi}{(\varkappa\sin 2p\alpha - p\sin 2\alpha)\cos p\pi}$$

$$\Delta = \sin 2p(\alpha-\beta) + p\sin 2(\alpha-\beta),\ \delta = \cos 2p(\alpha-\beta) - \cos 2(\alpha-\beta)$$

$$\varkappa_1 = (\varkappa+1)^2/2$$

where (ε_1, ε_2 are sufficiently small positive numbers). The solution of Eq. (2.6) is constructed in the same manner as the solution of the equations of the Winer-Hopf problems considered in /11/. We have ($\Gamma(z)$ is the gamma function)

$$\Phi^+(p) = -K^+(p)G^+(p)\left\{\frac{\tau_1}{p+1}\left[\frac{1}{K^+(p)G^+(p)} - \frac{1}{K^+(-1)G^+(-1)}\right] + \frac{\tau_2}{p+\lambda+1}\left[\frac{1}{K^+(p)G^+(p)} - \frac{1}{K^+(-\lambda-1)G^+(-\lambda-1)}\right]\right\} \tag{2.7}$$

$$(\operatorname{Re} p < 0)$$

$$\Phi^-(p) = \frac{pG^-(p)}{K^-(p)}\left[\frac{\tau_1}{(p+1)K^+(-1)G^+(-1)} + \frac{\tau_2}{(p+\lambda+1)K^+(-\lambda-1)G^+(-\lambda-1)}\right]\ (\operatorname{Re} p > 0)$$

$$\exp\left[\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\ln G(z)}{z-p}dz\right] = \begin{cases} G^+(p), & \operatorname{Re} p < 0\\ G^-(p), & \operatorname{Re} p > 0\end{cases}$$

$$K^{\pm}(p) = \Gamma(1 \mp p)/\Gamma(1/2 \mp p)$$

Using (2.7), the asymptotic forms (2.3) and a theorem of Abel-type /12/ we find

$$k_{II} = \frac{\sqrt{2}\,g\Gamma(\lambda+3/2)}{\Gamma(\lambda+2)G^+(-\lambda-1)}Cl^{\lambda+1/2} - \frac{\sqrt{\pi}}{\sqrt{2}\,G^+(-1)}\tau_1\sqrt{l} \tag{2.8}$$

We assume, that there are no stress concentrations at the head of the slip line. From (2.8) we have

$$l = D\left(\frac{|C|}{\tau_s}\right)^{-1/\lambda},\quad D = \left[\frac{2|g|\Gamma(\lambda+3/2)G^+(-1)}{\sqrt{\pi}\,\Gamma(\lambda+2)G^+(-\lambda-1)}\right]^{-1/\lambda} \tag{2.9}$$

Let us apply the principle of choice to find the angle of inclination of the slip line with respect to the edge. Let us consider the function

$$I(C) = \int_0^l \langle u_r\rangle|_{\theta=0}\, dr$$

By the well-known formula of differentiation of integrals which depend on a parameter, with variable limits we have

$$I^{\cdot} = \langle u_r\rangle|_{r=l,\,\theta=0}\, l^{\cdot} + \int_0^l \langle u_r^{\cdot}\rangle|_{\theta=0}\, dr \tag{2.10}$$

Since $\langle u_r\rangle\big|_{\substack{r=l\\ \theta=0}} = 0$ it follows from (2.10) that

$$I^{\cdot} = \int_0^l \langle u_r^{\cdot}\rangle\,|_{\theta=0}\, dr \tag{2.11}$$

Using the equality

$$I = -4(1-\nu^2)\,E^{-1}\Phi^-(1)\,l^2$$

we obtain by (2.7), (2.9), (2.11)

$$W = Q(\alpha)\,W_0(\alpha,\beta)\,\frac{(1-\nu^2)\,|\,C\,|^{-2/\lambda-1}}{E\tau_s^{-2/\lambda-2}} \tag{2.12}$$

$$Q = \frac{\pi}{\lambda+2}\left[\frac{2\Gamma(\lambda+{}^3/_2)}{\sqrt{\pi}\,\Gamma(\lambda+2)}\right]^{-2/\lambda}, \quad W_0 = |\,g\,|^{-2/\lambda}\,\frac{[G^+(-1)]^{-2/\lambda-2}}{[G^+(-\lambda-1)]^{-2/\lambda}}$$

The dependence $W_0(\beta)$ is qualitatively represented in Fig.2. The dashed line corresponds to the case $\pi/2 < \alpha < \alpha_0$ and the solid line to the case $\alpha_0 < \alpha < \pi$ ($\alpha_0 \approx 11\,\pi/15$ is the single root of the equation $W_0[\beta_*(\alpha)] = W_0(0)$, β_* is the point of the maximum of the function $W_0(\beta)$). It appears that $W_0(\beta_*) > W_0(0)$ for $\pi/2 < \alpha < \alpha_0$ and $W_0(\beta_*) < W_0(0)$ for $\alpha_0 < \alpha < \pi$.

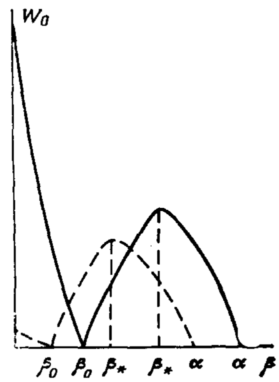

Fig.2

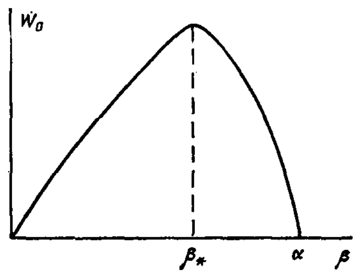

Fig.3

On the basis of the principle of choice for the original symmetric problem we can draw the following conclusions. For $\pi/2 < \alpha < \alpha_0$ the slip lines emerging from the corner point O are developing under the angle $\beta = \beta_*$ with respect to the edges, which increases as the wedge aperture angle increases. The values of the angle β_*, of the characteristic root λ and the coefficient D for some values of the angle α are given below ($\nu = 0.25$)

α, deg.	95	100	105	110	115	120	125	130
β_*, deg.	45	50	55	59	64	68	73	77
$-\lambda\cdot10^3$	28	54	80	106	134	162	191	221
$D\cdot10^5$	$0.860\cdot10^{-3}$	$0.238\cdot10^{-4}$	0.0329	1.45	19.3	102	343	855

For $\alpha_0 < \alpha < \pi$ the slip lines develop along the edges. The corresponding values of λ and D are

α, deg.	135	140	145	150	155	160	165	170	175
$-\lambda\cdot10^3$	252	284	315	346	375	403	430	455	479
$D\cdot10^3$	28.9	55.3	86.5	119	147	170	187	196	197

If the case $\alpha = \alpha_0$ is realized, then four slip lines emerge from the corner point, two of which are located along the edges and two are at an angle to the edges.

For $\alpha = \pi$ we arrive at the known result /4, 5/, where the direction of the development of the slip line is found from the condition of maximum of its length.

3. Let the case b) be realized. The boundary conditions differ from those paresented in paragraph 2 in that instead of the first condition (2.1) we have

$$\theta = \beta,\ \sigma_\theta = \tau_{r\theta} = 0\ (\beta \in\ ]0;\ \alpha[)$$

and the function $g > 0$ is determined by the formula

$$g=\frac{(2\pi)^\lambda}{2}\left[\lambda\sin\lambda(\alpha-\beta)-\frac{\lambda\sin\lambda\alpha}{\sin(\lambda+2)\alpha}\sin(\lambda+2)(\alpha-\beta)\right]$$

and $\lambda\in]-{}^1/_2;0[$ represents the single root of the characteristic equation

$$\sin 2(\lambda+1)\alpha+(\lambda+1)\sin 2\alpha=0$$

corresponding to the canonical singular problem in the interval $]-1;0[$.

The problem in question reduces to a Wiener-Hopf equation of the form (2.6), where

$$G(p)=\frac{[2\delta(\sin^2 p\beta-p^2\sin^2\beta)+\Delta(\sin 2p\beta+p\sin 2\beta)]\cos p\pi}{(\sin 2p\alpha+p\sin 2\alpha)\sin p\pi}$$
$$K(p)=-\operatorname{tg}p\pi$$

Using the solutions of this equation we obtain an expression for l and W, differing from (2.9) and (2.12) in, that

$$D=\left[\frac{\sqrt{\pi}\,g\Gamma(\lambda+2)G^+(-1)}{2(\lambda+1)\Gamma(\lambda+{}^3/_2)G^+(-\lambda-1)}\right]^{-1/\lambda},$$
$$Q=\left[\frac{16}{\pi(\lambda+2)}\right]\left[\frac{\sqrt{\pi}\,\Gamma(\lambda+2)}{2(\lambda+1)\Gamma(\lambda+{}^3/_2)}\right]^{-2/\lambda}$$

Fig.3 shows the function $W_0(\beta)$. The function reaches its maximum at the point β_*. The slip lines develop, in accordance with the principle of choice, at an angle β_* to the edges. The values of this angle, as well as the quantities λ and D, are given below

α, deg.	95	105	115	125	135	145	155	165	175
β_*, deg.	47	52	58	64	70	77	83	92	101
$-\lambda\cdot10^3$	100	248	348	414	456	480	493	499	499
$D\cdot10^3$	181	143	129	107	93	80	70	64	59

When $\alpha=\pi$, the formula for the length of the slip lines is indentical with the known formula /1, 6/ (the numerical factors differ by less than 0.001), and the angle of their inclination to the continuation of the crack is 76° (in /1/ this angle is found from the condition of maximum length of the slip line, and is equal to 72°).

4. In case c), the first condition of (2.1) is replaced by

$$\theta=\beta,\ \tau_{r\theta}=0,\ u_\theta=0\ (\beta\in]0;\alpha[)$$

and the root λ of characteristic equation of the corresponding canonical singular problem on the interval $]-1;0[$ is equal to $-2+\pi/\alpha$, while $g=\sin(\pi\beta/\alpha)$.

The Wiener-Hopf equation of the problem in question, the formulas for l and W, and the graph of the function W_0, have the same form as in the previous cases (see Sect.3) in which

$$G(p)=\frac{[\delta(\sin 2p\beta+p\sin 2\beta)+\Delta(\cos 2p\beta-\cos 2\beta)]\cos p\pi}{(\cos 2p\alpha-\cos 2\alpha)\sin p\pi}$$

The values of the coefficient D are given below

α, deg.	95	105	115	125	135	145	155	165	175
D	1.93	2.03	2.14	2.26	2.40	2.54	2.70	2.90	3.24

and the angle of inclination of the slip lines to the edges is $\beta_*\approx\alpha/2$, accurate to within 0.5°.

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ON A KELVIN PROBLEM*

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A problem of the stability of equilibrium of a system of interacting particles distributed within a bounded volume of Euclidean space is considered. Sufficient conditions for the instability and existence of the motions approaching the position of equilibrium without bounds, containing the Kelvin theorem /1/ as a special case, are obtained. The results are based on the general theory of instability of equilibrium in a force field with a subharmonic force function.

1. Let us consider the dynamics of a reversible system with kinetic energy $T = (g_{ij}v^iv^j)/2$ and force function $U(x)$. The motions are described by the Lagrange equations

$$(L'_{v^i})^{\cdot} - L'_{x^i} = 0, \quad v^i = d(x^i)/dt, \quad L = T + U, \quad i \leqslant n \tag{1.1}$$

The coefficients of the metric tensor g_{ij} and the function U are assumed to depend continuously on the x coordinates. We assume that the point $x = 0$ is critical for the force function U, and therefore $x = 0$ will represent the equilibrium of the system (1.1). We can assume that $U(0) = 0$. The function U will be called subharmonic if $\Delta U \geqslant 0$ where Δ is a Laplace-Beltrami operator taken with the minus sign:

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij} \frac{\partial}{\partial x^j}\right), \quad g = \det \| g_{ij} \|$$

It is clear that the condition of subharmonicity of the force function does not depend on the choice of the Lagrangian coordinates x^i.

Theorem 1. Let us assume that the force function U is subharmonic and its Maclaurin's series is different from zero. Then the equilibrium $x = 0$ will be unstable. In the analytic case the condition of subharmonicity is sufficient for the instability to occur.

Proof. Let g_0^{ij} be the values of the metric tensor at the point $x = 0$. We expand the force function U in a series in terms of homogeneous forms: $U_m + U_{m+1} + \ldots$, $m \geqslant 2$. It can be confirmed that $\Delta U = \Delta_0 U_m + \ldots$ where Δ_0 is the Laplace-Beltrami operator of the metric g_0^{ij} and repeated dots denote terms of order $\geqslant m - 2$. Since $\Delta U \geqslant 0$, we have $\Delta_0 U_m \geqslant 0$. The coefficients of the operator Δ_0 are independent of x, therefore the function U_m is subharmonic in the sense of the classical definition /2/.

Using the well-known inequality

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